THC White Paper

Local Volatilities Model

Examine Interest Rate Models for a Possible Negative Rate Regime

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Summary

On March 19, 2020, Treasury discount bill rates fell to 20 basis points, approaching negative rate scenarios, and a possible negative rate regime cannot any longer be ignored. Bankers should seek answers to the following crucial questions:

- How should the balance sheet be managed if negative rates prevail?
- Is the current risk management model appropriate for managing the balance sheet?
- How should Adjustable Rate Mortgages (ARMs) with caps and floors be priced? Are the embedded options (prepayment behavior) of a 30-year fixed mortgage loan affected?

This paper explains the Local Volatilities Model and provides empirical evidence of the validity and applications of the model.

The main contributions of the paper are:

- Demonstrate the applicability of the Local Volatilities model when interest rates are low or even negative
- Present the historical trends of interest rate term structure of volatilities and their applications
- Introduce the risk measure of rate distribution, in particular, the mean, variance and skewness of a rate distribution, which have many applications for balance sheet and risk management
- Highlight the limitation of interest rate models widely used today.

Historical data such as MIRF over a range of horizons and choice of curves, rates, and vol curves supporting the model are available at https://www.thomasho.com.

Keywords

Rate Distribution, forward rates, distribution skewness, arbitrage-free models, interest rate models, normal models, lognormal models, interest rate forecast, calibration, expectation hypothesis, Fisher Equation, Local Volatilities Model, term structure of volatilities

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Examine Interest Rate Models for a Possible Negative Rate Regime

A. Introduction: Modeling Interest Rate Movements in a Low Rate Regime

Challenges:

On March 19, 2020, Treasury discount bill rates fell to 20 basis points approaching the negative rate scenarios, and a possible negative rate regime cannot be ignored. Stakeholders of balance sheet management should raise these crucial questions:

- How should the balance sheet be managed if negative rates prevail?
- Is the current risk management model appropriate for managing the balance sheet?
- How should Adjustable Rate Mortgages (ARMs) with caps and floors be priced? Are the embedded options (prepayment behavior) of a 30-year fixed mortgage loan affected?

Interest Rate Models

I must first address the validity of the interest rate model. Interest rate models are central to valuing the embedded fixed-income options, which are prevalent in balance sheets1, and are essential to enterprise risk management and market-making. They are also the bedrock to building credit and liquidity models, as well as income/profitability simulations2. Financial economists cannot overstate the importance of interest rate models to our economic system.

The current low-interest-rate regime challenges the robustness of many operational interest rate models3. The purpose of this paper is two-fold:

- Explain the limitations of many operational interest rate models
- Describe how the Local Volatilities Model can provide the probability distributions of rates over any time horizon inferred from the swaption prices.

I will discuss how the Local Volatilities model addresses these questions. In the discussion, I will introduce new approaches to balance sheet total return, lending, and funding strategies. Extending Ho and Mudavenhu [7] that presented the Local Volatilities model and its applications in 2019, the purpose of this paper is to provide an intuitive explanation of the model and the implied probability distributions of interest rates based on a sample of months from March 2019 to February 2020. From the empirical results, I will discuss several balance sheet strategies.

Furthermore, the current low-interest-rate regime has shaken financial economists to rethink many of the fundamental practices in the capital markets. Is the nominal interest rate the sum of the real economic return and the inflation rate (Fisher Equation)? Can the shape of the yield curve predict

[1] Reference White Paper "Embedded Option Pricing"
[2] Reference White Paper "pOAS: A New Measure of Profitability"
[3] Reference White Paper "Negative Interest Rates and Interest Rate Models"

interest rates (Expectation Hypothesis) [4]? Are operational interest rate models valid when underlying economic assumptions are in question? Some media reports echo these questions: Kochkodin, Bloomberg News, September 2019, reported that "negative interest rates broke the Black Scholes model, Pillar of Modern Finance." [11] Gunjan Banerji, 10/17/2019, Wall Street Journal reported "Negative U.S Interest Rates? Option Traders Say Yes." The urgency to re-evaluate interest rate models is critically apparent.

Ho-Lee Model

Ho-Lee [5] introduced the arbitrage-free interest rate model in 1984. "Arbitrage-free" means that the evolution of the yield curve movements must be consistent with the valuation of capital market benchmark derivatives and the market discount curve. Swaptions and the swap curves are examples of derivatives and market discount curves, respectively. I called the process of implying the underlying yield curve movements from the derivative prices "Calibration." [12] Since the derivative prices determine the yield curve movements by calibration, by definition, there is no arbitrage opportunity in the capital market to exploit the pricing of derivatives. The Ho-Lee result shows that the calibrated probability distributions of interest rates can take on many forms depending on the time horizon and the selected rate to be considered.

Normal Model versus Lognormal Model

When interest rates were high, around 6%, in the 1980s, interest rates tended to rise and fall by the same amount. The Normal Model [5] [10] hypothesizes that rates are equally likely to rise and fall, which is approximately consistent with the swaptions' prices. That is, the normal model measures the rate change by the basis point change, a basis point shift. However, the Normal Model allows for negative interest rates, an assumption that financial economists categorically rejected. In response, the financial modelers replaced the Normal Model by the Lognormal Model.

The Lognormal Model [1] [2] [3] assumes interest rates take random walks, analogous to that of the stocks, which always remain positive. The model measures changes by the percentage change. When rates are low, the percentage of change can be high. The Lognormal Model projects rates rising rapidly, creating a positive skewness of the rate distribution, an undesirable attribute.

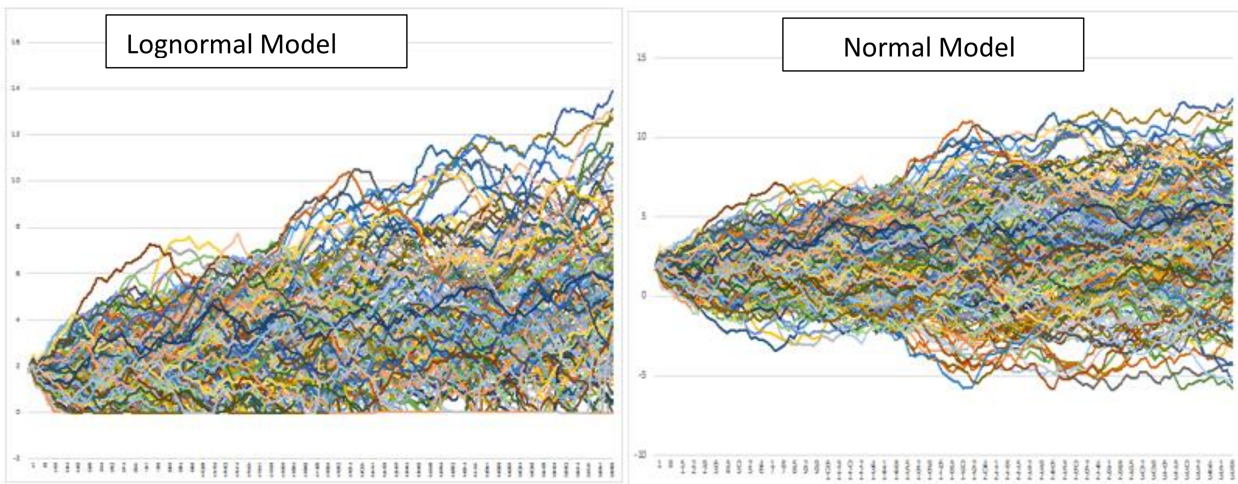
Nearly 35 years of research sought to resolve these inconsistencies with observed market prices. The figures below illustrate the complexity of interest rate simulations by comparing the outputs of a Lognormal Model and a Normal Model. The charts depict the monthly one-month swap rates simulated over 30 years. The BGM Model[4], an operational interest rate model, generates 257 interest rate paths under a Lognormal (on the left) and Normal Model (on the right). These interest rate paths are consistent with market pricing of the swaps and a portfolio of swaptions, satisfying the arbitrage-free model assumption, as explained in THC White Paper [1].

[4] Reference: Technical Notes explains the BGM model

The simulation shows that the Lognormal Model does not allow negative interest rates, resulting in a positively skewed rate distribution. By way of contrast, the Normal Model allows for significantly negative interest rates while maintaining the rate distribution to be symmetric.

Figure 1 underscores the limitation of these interest rate models. Today, interest rate frequency distribution is neither lognormal nor normal. The former does not accept negative rates. The latter cannot take out-of-money option prices to determine the asymmetric cap and floor premiums and often leads to significant negative interest rates.

Figure 1. Interest rate paths over 30 years. The results depict the limitations of the lognormal and normal interest rate models. The Lognormal Model is positively skewed with the mean one-month rate exceeds the median rate at a time horizon. The Normal Model has no skewness with the average rate and the median rate the same.

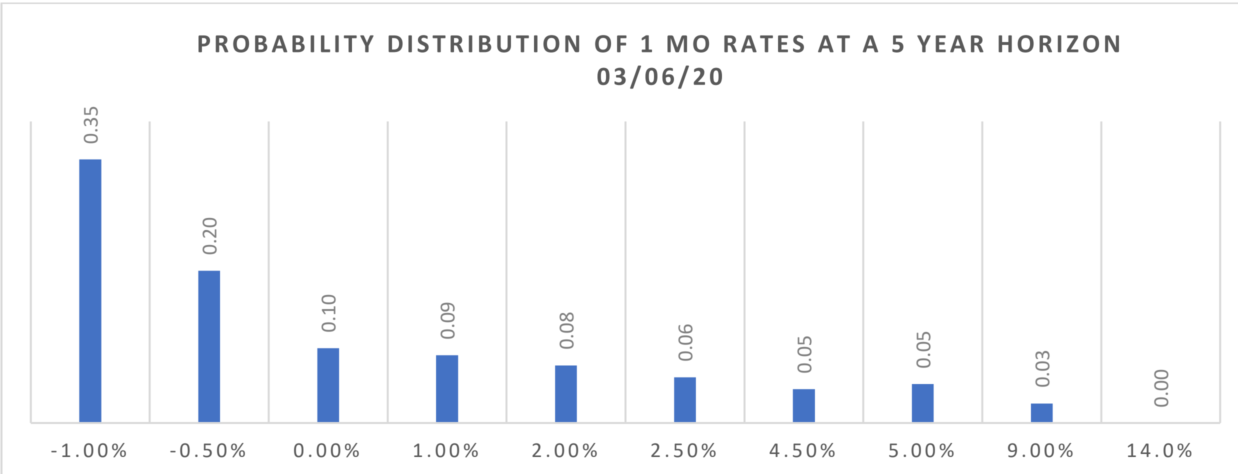


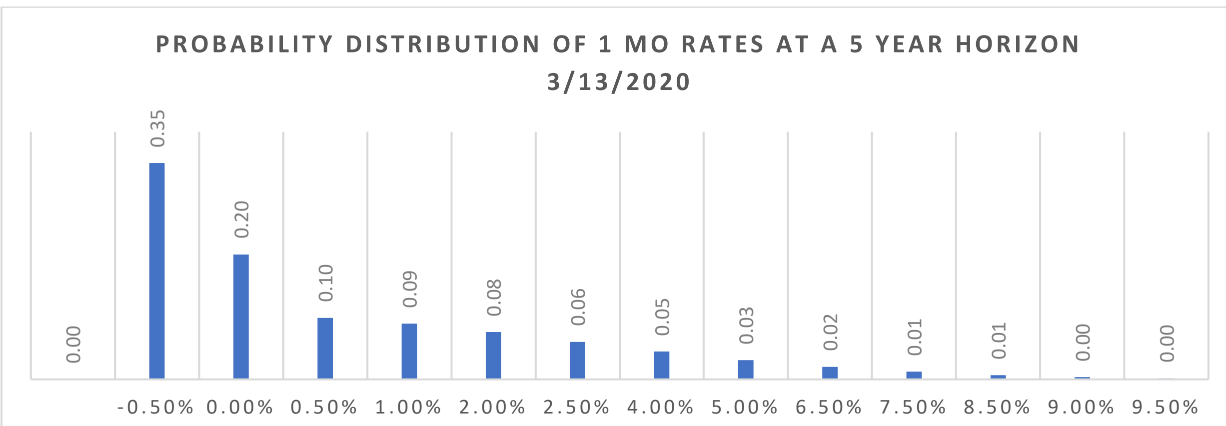
Local Volatilities Model

Interest Rate Models should accept negative interest rates implied by swaption prices and asymmetric distribution of the frequency distribution of interest rates. The Local Volatilities Model [7] [8] [9] follows the Ho-Lee formulation by not assuming a specific interest rate probability distribution, such as the normal or lognormal distributions. Instead, the Local Volatilities Model assumes the volatility is local, depending on the time horizon and the projected rate levels, local in the projected time, and rate level.

The model calibration uses both at-the-money and out-of-the-money swaptions to calibrate the interest rate movements and hence, the rate distributions. The Local Volatilities model uses the 80 at-the-money swaptions with terms 1, 2, 3, 4, 5, 7, and 10 years and tenor 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 30 and 91 out-of-the-money swaptions with rates -100, -50, 50, and 100 bp, for the same term and tenor. The Local Volatilities Model can then present the underlying rate distributions. In this sense, the Local Volatilities Model is arbitrage-free, as it is consistent with the capital market pricing of the yield curve and observed swaption prices.

Figure 2 below depicts the one-month Treasury rate distribution over a five-year horizon as of March 6, 2020, and March 13, 2020, when the Coronavirus pandemic greatly affected the Treasury and swaption markets.





The results show that there is a significant implied probability of negative interest rates prevailing in 5 years. Indeed, the results indicate there is a 35% and 20% chance the one-month rate will reach less than -0.5% and fall between -0.5 and 0%, respectively.

Three risk drivers are affecting the rate probability distribution: (1) the forward rate (2) market perceived interest rate uncertainty, called the Volatility Curve, and (3) the out-of-the-money swaptions. These risk drivers determine the median, variance, and skewness of the distribution.

The following section describes the historical trends of the rate distributions as affected by these three risk drivers.

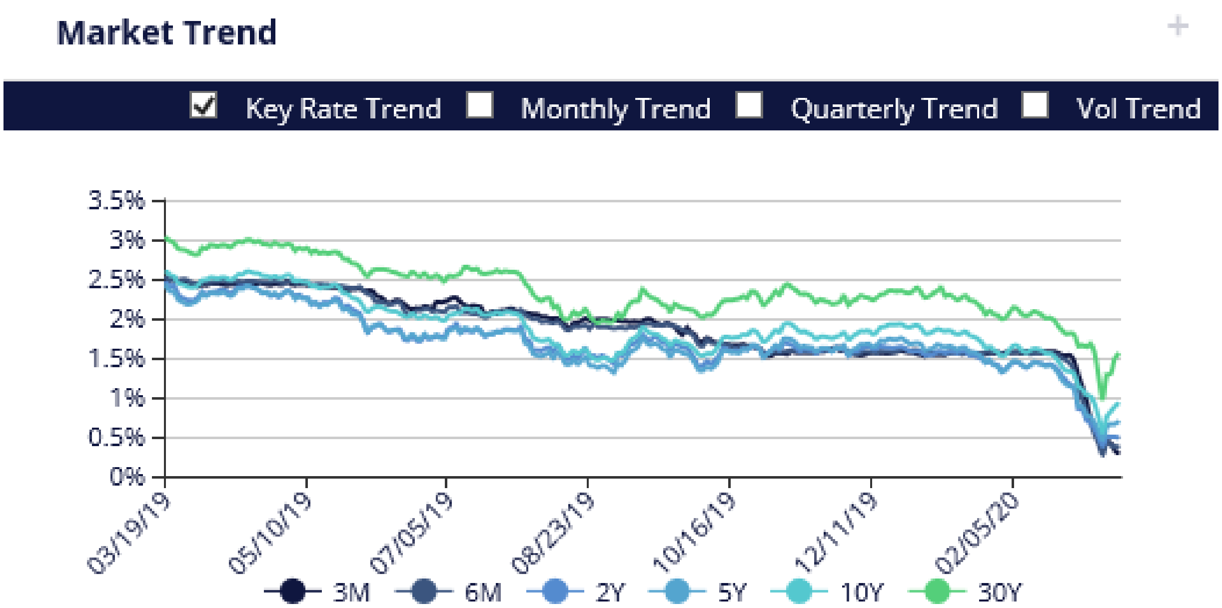
B. Rate Distribution Trend Analysis

Market Conditions

The shape of the yield curve determines the forward rates. The historical trends of the key rates, shown below, can partly explain the median rate of the historical rate distributions.

Figure 3. The Historical Trends of Treasury Key Rates.

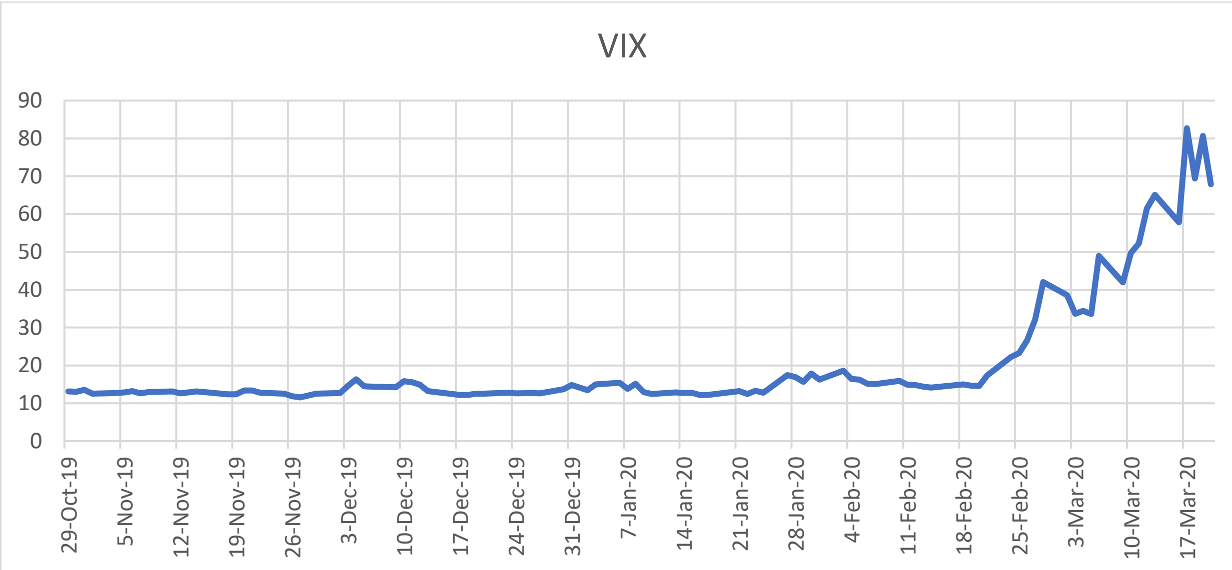
The 12- month daily Treasury key rates depict the dynamic nature of the rate level and the shape of the yield curve. Via the swaption market, the perceived uncertainty and skewness of the rate distribution can provide useful information on a forward-looking basis.



Volatility Curve

Before I discuss how to measure interest rate uncertainty, I will discuss the popular use of measuring stock market uncertainty with VIX, the ticker symbol for Chicago Board Options Exchange's CBOE Volatility Index. S&P 500 index option prices are used to measure the stock market's expectation of volatility. The VIX value is derived from the Black-Scholes option pricing model using the S&P index options expiring in 30 days. The current VIX index value quotes the expected annualized change of the S&P Index. The graph below depicts the dramatic increase in market uncertainty because of the coronavirus pandemic. Note that the Black-Scholes model is lognormal, and the uncertainty is measured by the percentage change in value. As the S&P index falls, the percentage change can increase even if the risk in price change remains that same. The difference in the measure of uncertainty using the change in price, as opposed to percentage, is particularly crucial for interest rate risk measure of uncertainty.

Figure 4 Historical Trends of VIX. VIX is the market's perceived uncertainties. VIX has shown to be useful to infer the market consensus on risk lying ahead.



THC models are used to measure the interest rate uncertainty using the analogous, but more complex methodology, because the rate volatilities are vastly different from stock price volatility. The reasons are:

1. Yield curve movements depend on the yield curve rates, not one stock price.

2. Yield volatility is related to price volatility, depending on the bond/swap terms and conditions

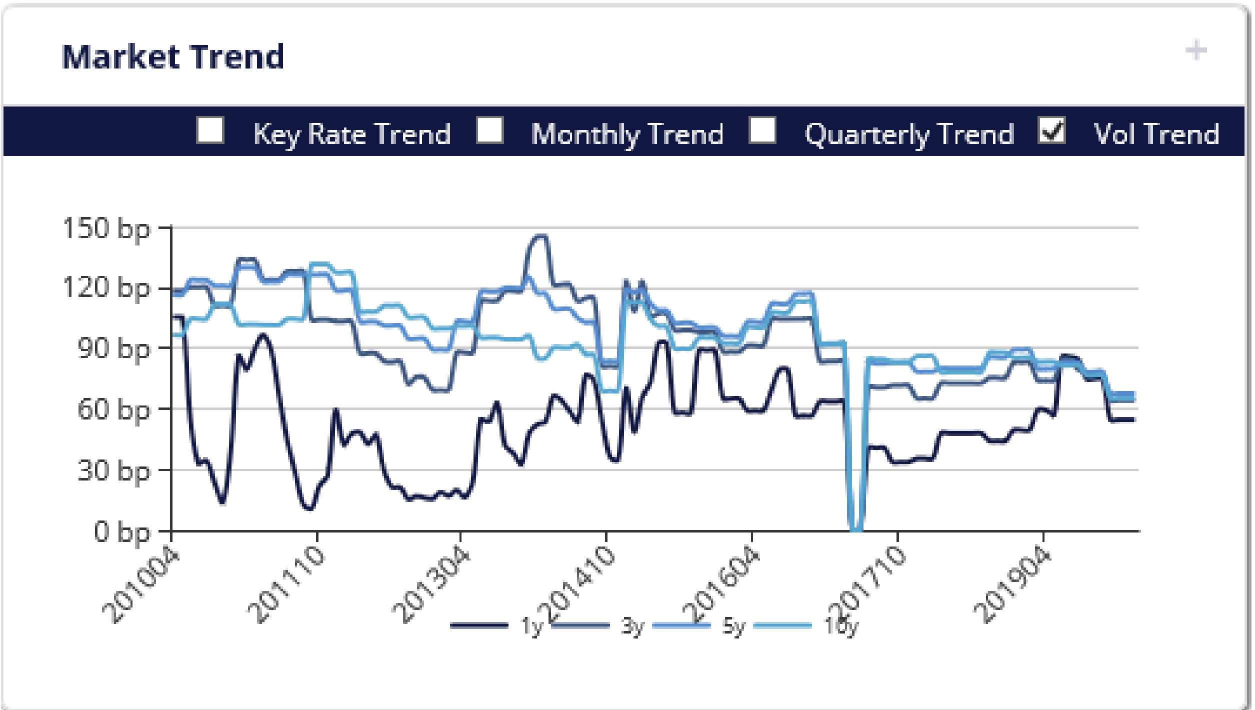
3. Yield volatility can be measured by the percentage change of basis point. But when the yield is negative, the proportional change cannot apply. For example, if the volatility is 20% and rates are -1%, then the rates would be either -1.2 or -0.8. If the rate is -5%, then the rates would be either -1% or -6%. That means the lower the negative rate, the higher the variance.

The volatility curve historical trend was calculated using the Local Volatilities Model. Because of reason (1), I provide the term structure of volatilities to present the volatility for each term. For a reason (2), I use the Local Volatilities model to relate the price changes to the yield changes for the instruments along the yield curve. For reason (3), I use the basis point change to avoid the proportional change when rates are low, or even negative.

Therefore, analogous to the VIX definition, the rate volatility is defined as the one standard deviation of the probability distribution of a chosen rate measured in an annualized basis point shift over a specified time horizon.

Figure 5. shows the market’s perceived rate uncertainty. As expected, the market volatility was high in post-2008 financial crisis, and again when the Brexit event occurred in 2011. The volatility again rose in the period of the trade war and more currently during the coronavirus pandemic.

Figure 5. Historical Volatility Curves on interest rate risks. The results show that the short-term uncertainty is more pronounced in this coronavirus pandemic period, as reflected in the rise of the short-term rate volatilities. Note that while recent volatilities are lower than those in 2010, the reason is that the interest rates have fallen substantially since 2010. If the interest rate risk is measured by percentage change (lognormal model), then the current volatility is over 1500%.



Negative Rates

Given the fall in interest rates, increased volatility, and the change in the skewness of the distribution, the implied probabilities of negative interest rates change over the sample period. Below is the monthly estimate of the rate distributions over 12 months starting at the end of March 2019. The results show that August 2019 and March 2020 have significant implied negative rates.

Figure 5. Historical Implied Probabilities of Negative Interest Rates. While most periods do not exhibit a positive probability of negative interest rates, the swaption markets identify the likelihood of negative rates in stress periods.

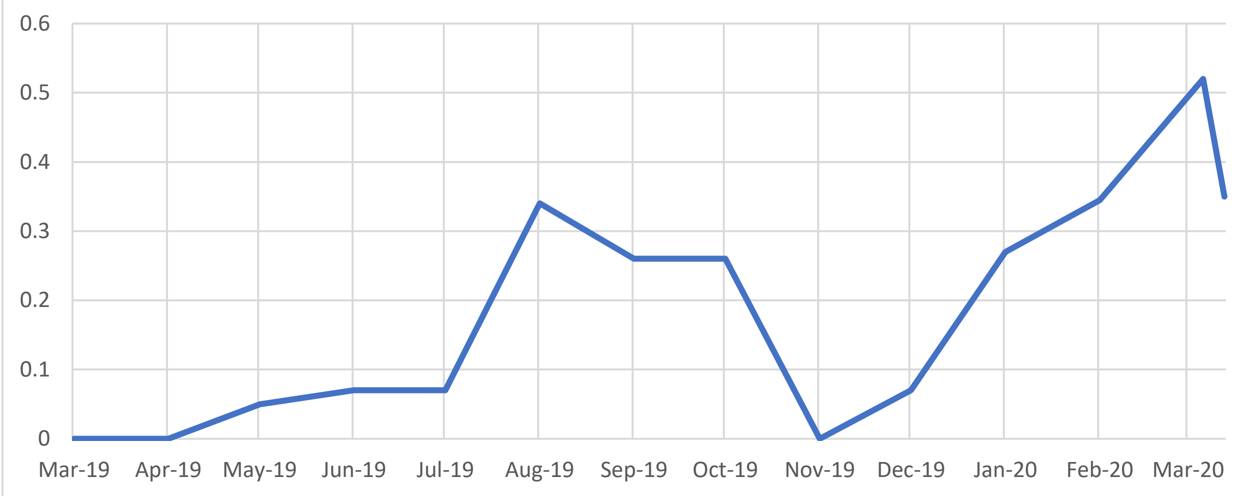


Figure 5 Probability of Negative Interest Rate

The most substantial one-day losses of 2019 occurred in August after we saw significant market uncertainty as China suggested using their currency to strategize the trade war. Still, the risk subsided after the trade discussion resumed in the following months.

By December, the coronavirus outbreak in China became widespread, resulting in a rise in global market performance uncertainty. The risk heightened when the spread of the virus became a pandemic in March. Comparing the vol curve and rate trends, the significant rise in the probability of negative rate is mainly due to the drop in the rate level, not as much as the increase in the normal rate risk.

C. Historical Rate Distributions for Sample Dates between March 2019 to March 2020

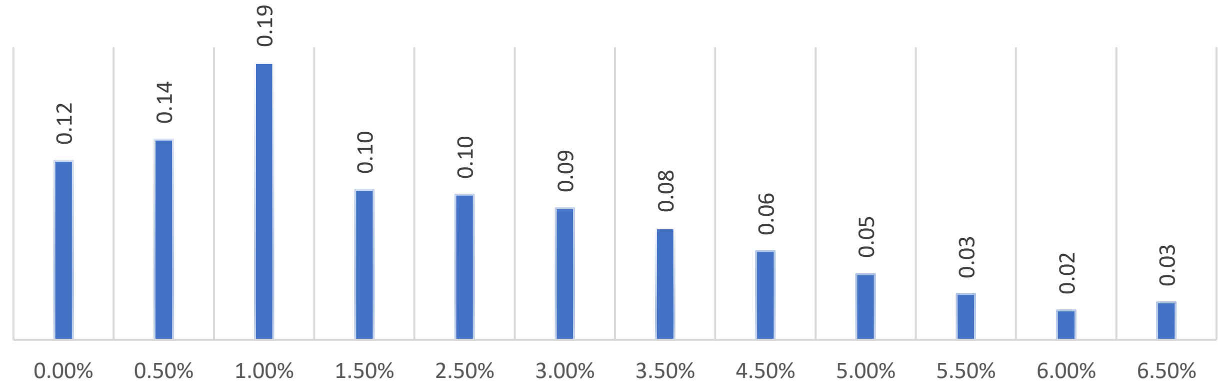
On March 29, 2019, the Treasury curve was relatively flat, suggesting the projected forward rates were not high, and the curve level was high relative to subsequent months. The implied volatility was also low before the increase in volatility that resulted from the Trade War uncertainty.

Empirical Evidence

The results are reported in Figures 3, 4, 5, and 6 show that the implied distribution is consistent with these observations. The distribution model is around 1%, with the rates not falling below zero. As a result, the probability distribution is positively skewed, with a median of 2%.

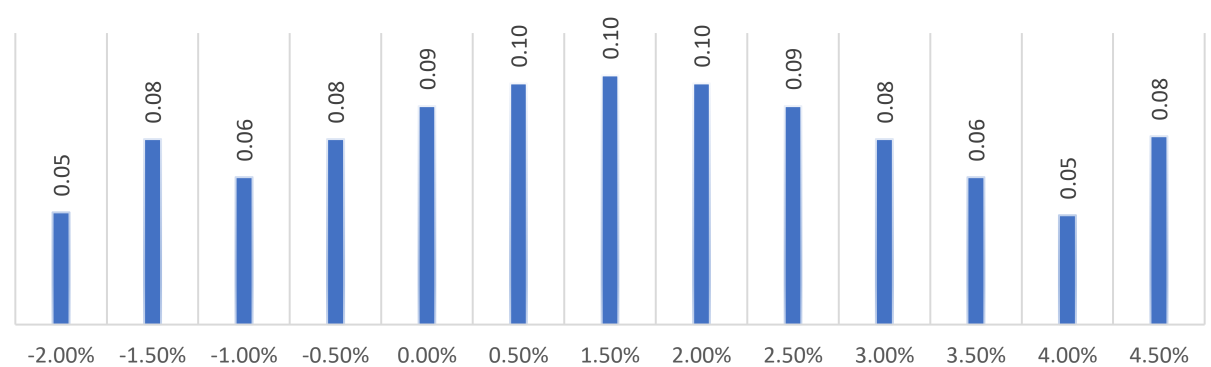
The following Figures 6, 7, 8, and 9 further support these observations based on additional sample data.

FIGURE 6 PROBABILITY DISTRIBUTION OF 1 MO RATES AT A 5 YEAR HORIZON 03/29/2019



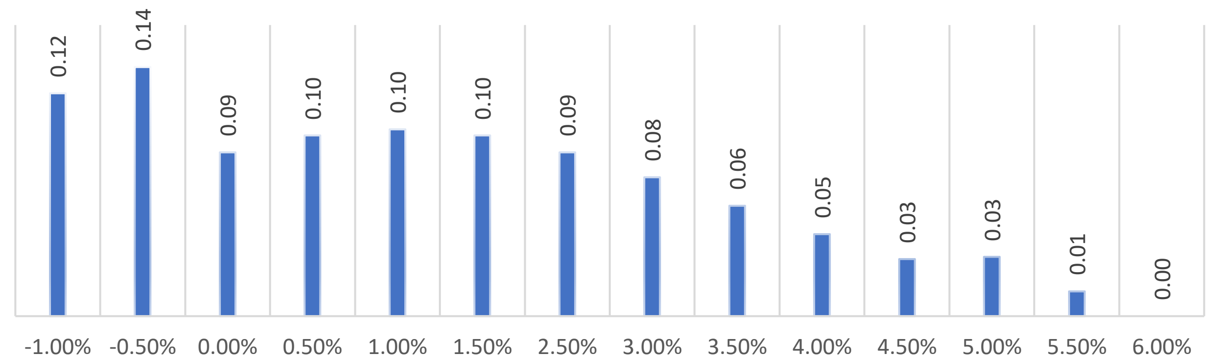
As explained above, because of the escalation of the Trade War, August 2019 experienced significant market uncertainty. Figure 3 shows that interest rates rose significantly, and Figure 4 shows that the implied volatility increased in the same period. Figure 6 shows that the median of the distribution shifted higher, and the rates fell below zero.

FIGURE 7 PROBABILITY DISTRIBUTION OF 1 MO RATES AT A 5 YEAR HORIZON 09/30/2019



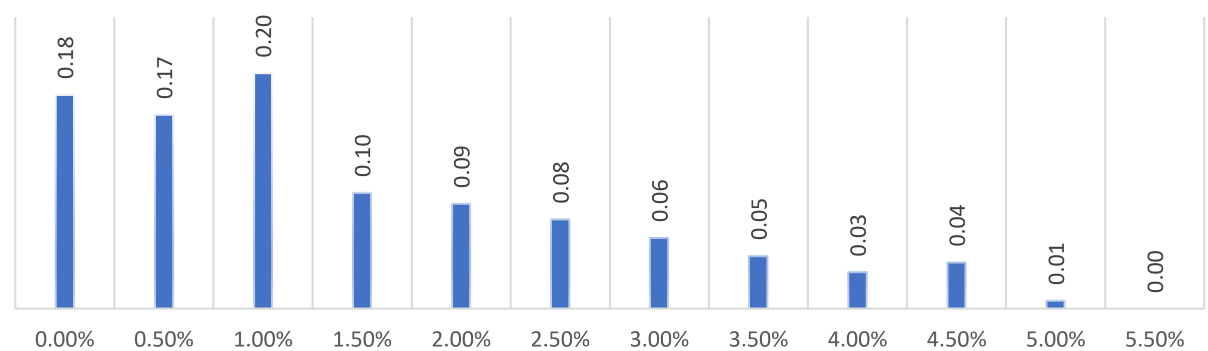
In late October, the yield curve is no longer inverted, and volatility fell. Figure 8 shows that the rate of distribution has a lower probability of negative rates, and the skewness is more pronounced than in the previous month.

FIGURE 8 PROBABILITY DISTRIBUTION OF 1 MO RATES AT A 5 YEAR HORIZON 10/31/2019



By late November, the volatility fell further, and the yield curve remained flat around 1.75%. Figure 8 shows that the rate distribution does not have negative rates, and the skewness remained positive.

FIGURE 9 PROBABILITY DISTRIBUTION OF 1 MO RATES AT A 5 YEAR HORIZON 11/29/2019



The results show that

- The dynamic nature of the distribution indicates that the median, variance, and skewness of the distributions continually adjust to the market conditions as the yield curve, implied volatilities of the at-the-money swaptions, and the skewness affected by the out-of-the-money swaption pricing change over time.
- A lognormal distribution cannot model the distribution because lognormal distributions do not accept negative interest rates. When the interest rate is only 0.2%, the rate distribution is constrained to a narrow band on variations. Therefore, the projected simulations cannot be consistent with the swaption prices.

- The normal distribution cannot be consistent with the current low-interest-rate scenario as a result of the market "Risk Arbitrage" practice. If rates become significantly negative, investors have alternative investments without paying a significant price premium that results in high negative interest rates. For these reasons, the historical estimated distributions tend to be positively skewed, and there is a limit to the level of negative rates.

Summary

In summary, the results show that the Rate Distribution is dynamic, continually adjusting to the market perception of interest rate uncertainties and the skewness of the distribution. Many operating interest rate models estimate the skewness and floor rate separated from the model calibration estimation process[5]. Additionally, some interest rate models are calibrated to at-the-money options only, excluding out-of-the-money options. The next section will discuss the limitations in fixed-income pricing when the extent of projected negative rates and rate distribution skewness is not dynamically estimated.

D. Implications of Skewness: Mean and Median of Rate Distribution

The previous sections show the importance of evaluating the rate distributions and the dynamic nature of the rate distribution, neither confined to a normal or lognormal distribution. The distributions should be defined by the mean, variance, and skewness.

Forward Rates and Expected Rates

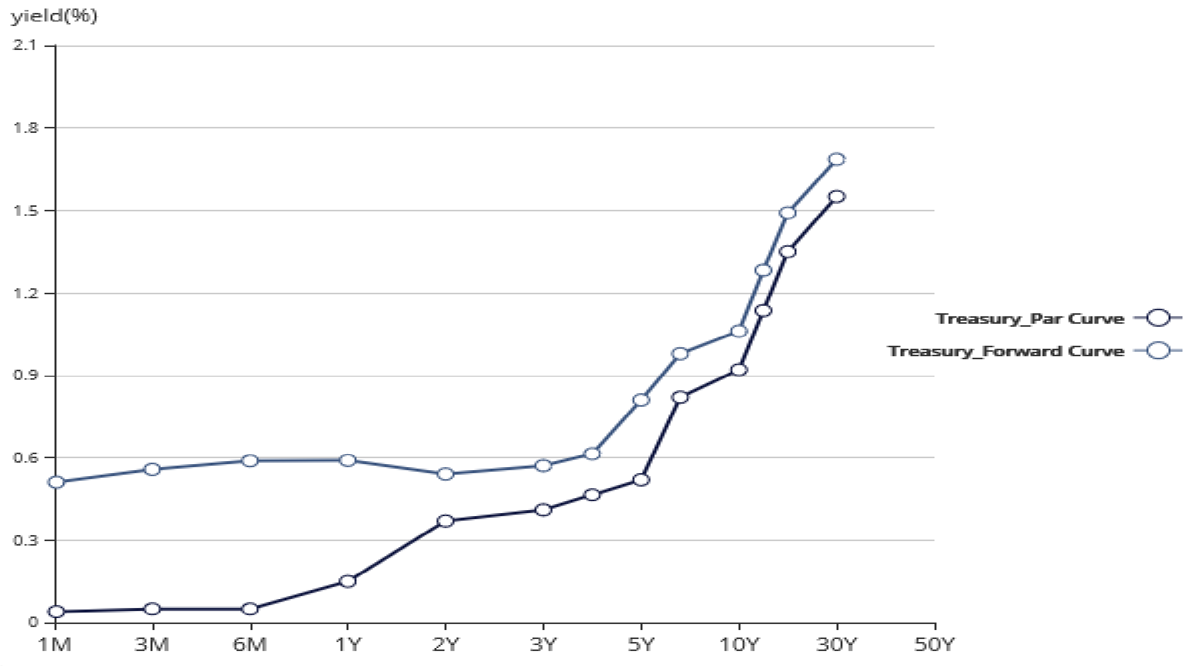
Rate Distribution skewness concludes that the forward rates, as determined by the yield curve, are not the market's expected rates. The forward rates and expected rates are different, contrary to the Expectation Hypothesis. The Rate Distribution can explain the poor performance of the forward rates as a rate forecasting tool and offer an alternative to measuring the market expectation of future rates.

A market scenario can best explain the importance of using Rate Distribution. On March 20, 2020, the yield curve was upward sloping. The one-month and 10-year rates were 0.04% and 0.92%, respectively. However, the capital market did not have to believe the one-month rate would rise from 0.04% to 0.51%, or the ten year-rate from 0.92% to 1.06%, in one year. Instead, the capital market traded the caps to reflect the market rate expectation, creating a negative skewness in the rate distribution. At the same time, the Federal Reserve Bank affected a steep yield curve for a broader economic purpose.

Figure 10 presents the difference between the par curve and the forward curve and the relevance of the Market Interest Rate Forecast (MIRF)

Treasury as of 03/20/2020															
Time			1m	3m	6m	1y	2y	3y	4y	5y	7y	10y	15y	20y	30y
Par Curve	View Line:	☑	0.04000	0.05000	0.05000	0.15000	0.37000	0.41000	0.46500	0.52000	0.82000	0.92000	1.13500	1.35000	1.55000
Spot Curve	View Line:	☐	0.03995	0.05013	0.04945	0.15097	0.37042	0.41054	0.46571	0.52144	0.82759	0.92926	1.15612	1.39398	1.62141
Forward Curve	View Line:	☑	0.51134	0.55738	0.58842	0.59011	0.54046	0.57073	0.61417	0.81003	0.97826	1.06069	1.28171	1.49090	1.68578

[5] Calibration is explained in the White Paper 2019 "Embedded Option Pricing".



The average of Rate Distribution, as depicted in Figure 2, represents the market expectation, which is not the same as the forward rates, which intuitively is the median of the distribution.

The Skewness in Loan Pricing

The depiction of the rate distribution has many applications in the capital market. The model provides an estimation of the value of caps and floors of adjustable-rate instruments as the rate distribution reports the probabilities of becoming in-the-money for the caps and floors. Consider the following example:

> *A 30-year Adjustable Rate Mortgage (ARM) may have an interest rate structure 2/2/5, indexed to a 1-year Treasury rate with a margin of 2.2%. If the loan has an initial rate of 3.25%, then the cap at the reset would be useful when the one-year Treasury rises to 5.25 (=3.25%+2%), or the one-year Treasury rate increases from today's rate of 0.8% to 3.25% in 5 years. The rate distribution provides the probability of the interest to reach the cap at 5.25%. If that scenario is not likely to prevail, the lending officer could lower the initial rate so that the ARM is competitive to the rate of the 30-year fixed-rate mortgage.*

Stress Test the Balance Sheet and Limitations of Current Interest Rate Models

The distributions also provide valuable information for stress testing the balance sheet or a portfolio of balance sheet instruments. The results provide some measure of the likelihood of a stress scenario that may occur. The "likelihood" is estimated from capital market prices and is, therefore, objective and transparent. The Local Volatilities Model can simulate the balance sheet under the downshift of the yield curve for stress testing purposes.

Researchers developed many interest rate models in the 1980s when interest rates were much higher than those of today.

- The lognormal model cannot accept negative interest rates, and therefore, will fail for the stress test
- The normal model accepts negative interest rates, but the projected model interest rates could be significantly negative. However, rates cannot be significantly negative when employing the risk arbitrage activities of "putting money under the carpet." Any assumption of flooring the rate at a particular level would violate the arbitrage-free model assumption. That is, such a model cannot be consistent with the basic discount cash flow model when the embedded option value is negligible and hence would misprice the balance sheet items.
- Another commonly used model, the Displaced Diffusion model, specifies the minimum interest rate based on a lognormal model. Empirical evidence has shown that the minimum rate is dynamic, and the floor rate has to be estimated by the calibration.

Balance Sheet Enhance Return Strategies

The Local Volatilities model has introduced the concept of rate probability distribution. From the rate probability distribution, interest rate strategists can manage with mean, variance, and skewness of the probability distributions. To date, controlling the rate level risk and interest rate volatility is widely used in practice. In the low-interest rate regime, balance sheet strategies should also focus on the skewness.

Specifically, the pricing of caps and floors, and the use of out-ot-the-money options are essential for hedging, buying, or selling options. There are many out-of-the-money options sold to customers on the balance sheet. The value of skewness of the interest rate probability distribution can be perceived differently from those priced in the capital market. The variance in the price offers profit opportunities.

E. Conclusions

Financial economists have categorically rejected the possibility of negative interest rates for over 35 years, resulting in much-unwarranted research and the development of interest rate models. Today, the low-interest-rate regime has "broke(n) the Black Scholes model, Pillar of Modern Finance." Economists should critically evaluate the current operational interest rate models.

This paper shows:

- Distribution Skewness. The interest rate model should use both capital market pricing of At-the-Money and Out-of-the-Money options to define the Rate Distribution, the projected interest rate minimum, and the distribution skewness. The skewness affects the pricing of caps and floors of an adjustable-rate instrument. Furthermore, interest rate models without the calibration of the interest rate distribution do not provide a valuation model consistent with capital market derivative markets.
- The Expectation Hypothesis. The interest rate model's Rate Distribution determines the market expected interest rate level, which is a more appropriate rate forecast than that suggested by the Forward Curve, as the Expectation Hypothesis suggests.

- The Fisher Equation. The concept of the nominal interest rate is the sum of the economy's real rate of return, plus the inflation rate (the Fisher Equation) is a positive theory. By way of contrast, the arbitrage-free model is a normative theory. The arbitrage-free model can lead to timely actionable decisions and not relying on the concept of general market equilibrium.

The main contributions of the paper are:

- Demonstrate the applicability of the Local Volatilities model when interest rates are low or even negative
- Present the historical trends of interest rate term structure of volatilities and their applications
- Introduce the risk measure of rate distribution, in particular, the mean, variance and skewness of a rate distribution, which can have many applications for balance sheet and risk management
- Highlight the limitation of interest rate models widely used today.

Appendix

Financial economists have made tremendous progress in interest rate modeling in the past 35 years. Models such as Ho-Lee, Black-Derman-Toy, Cox-Ross-Ingersoll, Hull-White, Black-Karasinski, Brace-Gatarek-Musiela, and Longstaff-Santa-Clara-Schwartz continually enhance interest rate modeling. These models have also introduced many new concepts such as martingale, lattice, recombining, delta hedge, risk-neutral measure q, physical measure p, kernel, vol surface, lognormal & normal models, OAS and Greeks, string theory, stratified sampling, rational option exercise rules, and CEV skew model. The study of interest rate modeling has even become a core course in the mathematics department.

For the following equations, I use the notations:

r = a short-term rate

∅= adjustment factor in ensuring the interest rate movements are arbitrage-free

$\sigma(t)$ = term structure of volatilities

dz= wiener process; normal distribution over a short time

The basic models from which extend many other models for credit risks, yield curve movements, computational efficiencies:

$$dr = \emptyset(t)dt + \sigma(t)dz$$

The normal model where the term structure is independent of the rate level. The Heath-Jarrow-Morton (HJM) models determine the term structure of volatilities using the forward rates.

$$dr = \emptyset(t)rdt + \sigma(t)rdz$$

The lognormal model, where the term structure of volatilities measures the proportional change of rates. Black-Derman-Toy (BDT) avoids negative rates.

$$dr = \emptyset(t)(\, l - r)dt + \ \sigma(t)r^{0.5}dz$$

The Cox – Ross – Ingersoll (CIR) model where the rates mean revert to some long-term rate and rate change depends on the term structure of volatilities and positively related to the rate level

$$dr = \emptyset(t, r)dt + \ \sigma(t)r^{\alpha}dz$$

Constant Elasticity Variance model where elasticity α can be between 1 and 0.1. BGM (LIBOR Market Model) is a CEV model with an efficient algorithm that fits precisely to market pricing of swaptions. These models do not accept negative rates. To allow for negative rates, BGM and Hull-White models allow for a modeler to set the floor rate, which can be negative.

$$dr = \emptyset(t)dt + \ \sigma(t)\eta(r)dz$$

Local Volatility Model allows for the out-of-money options to specify the rate distribution enabling the rate distribution $\eta(r)$ to fit the out-of-the-money option prices.

The established interest rate models do not allow for the swaption prices to specify the rate distribution. Instead, these models impose the swaptions be priced based on a form of lognormal models or a normal model with a constraint of the minimum rate. The Local Volatility Model seeks to overcome these problems in a low-interest rate regime by explicitly determine the rate distribution $\eta(r)$ from the OTM option prices.

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